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On multi-point boundary value problems for linear ordinary differential equations with singularities ☆

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1. Introduction

Suppose $n \geqslant 2$ is an arbitrary natural number, $-\infty < a < b < +\infty$, and $p_i :]a, b[\to \mathbb{R}$ $(i = 1, \dots, n)$ and $q :]a, b[\to \mathbb{R}$ are measurable functions. The differential equation

$$u^{(n)} = \sum_{i=1}^{n} p_i(t)u^{(i-1)} + q(t) \tag{1.1}$$

is said to be *singular* if some of its coefficients are nonintegrable on $[a, b]$ having singularities at one or several points in this segment. For the singular equation (1.1) two-point boundary value problems and multi-point problems of the Vallée-Poussin and Cauchy–Nicoletti types have been investigated more or less in detail (see [1,2,5–9,11–13,21–25,27] and the references therein). As for so-called nonlocal multi-point problems (i.e., problems with conditions connecting the values of a desired solution and its derivatives at different points in the segment $[a, b]$), are studied mainly for a second order equation (see, e.g., [10,

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14–20]), while for higher order equations these problems only rarely studied. The present paper is devoted to the investigation of two such problems. More precisely, we consider the singular equation (1.1) with the boundary conditions

$$u^{(i-1)}(t_0)=0 \quad (i=1,\dots,n-1),$$
$$\sum_{j=1}^{n-n_1}\alpha_{1j}u^{(j-1)}(t_{1j})+\sum_{j=1}^{n-n_2}\alpha_{2j}u^{(j-1)}(t_{2j})=0, \tag{1.2}$$

or

$$u^{(i-1)}(a)=0 \quad (i=1,\dots,n-1), \qquad \sum_{j=1}^{n-n_0}\alpha_j u^{(j-1)}(t_j)=0, \tag{1.3}$$

where $n_k\in\{0,\dots,n-1\}$ $(k=0,1,2)$,

$$a<t_0<b, \qquad a\leqslant t_{1j}<t_0\ (j=1,\dots,n-n_1),$$
$$t_0<t_{2j}\leqslant b \quad (j=1,\dots,n-n_2), \qquad a<t_j\leqslant b \quad (j=1,\dots,n-n_0),$$

and by $u^{(j-1)}(a)$ (by $u^{(j-1)}(b)$) it is understood the right (the left) limit of the function $u^{(j-1)}$ at the point a (at the point b).

Throughout the paper, we use the following notations:

- $\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$.
- $[x]_+$ and $[x]_-$ are the positive and the negative parts of the number x, i.e., $[x]_+=\frac{1}{2}(|x|+x)$, $[x]_-=\frac{1}{2}(|x|-x)$.
- $C^{n-1}_{n_1,n_2}(]a,b[)$ is the Banach space of $(n-1)$-times continuously differentiable functions $u:]a,b[\to\mathbb{R}$ having the limits

  $$\lim_{t\to a}(t-a)^{n_{1i}}u^{(i-1)}(t), \qquad \lim_{t\to b}(b-t)^{n_{2i}}u^{(i-1)}(t) \quad (i=1,\dots,n),$$

  where

  $$n_{1i}=[i+n_1-n]_+, \qquad n_{2i}=[i+n_2-n]_+ \quad (i=1,\dots,n). \tag{1.4}$$

- The norm of an arbitrary element u of this space is defined by the equality

  $$\|u\|_{C^{n-1}_{n_1,n_2}}=\sup\left\{\sum_{k=1}^{n}(t-a)^{n_{1i}}(b-t)^{n_{2i}}\left|u^{(i-1)}(t)\right|:\ a<t<b\right\}.$$

- $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ is the space of functions $u\in C^{n-1}_{n_1,n_2}(]a,b[)$ for which $u^{(n-1)}$ is locally absolutely continuous on $]a,b[$, i.e., absolutely continuous on $[a+\varepsilon,b-\varepsilon]$ for an arbitrarily small positive ε.
- $L_{n_1,n_2}(]a,b[)$ is the Banach space of integrable with the weight $(t-a)^{n_1}(b-t)^{n_2}$ functions $q:]a,b[\to\mathbb{R}$ with the norm

  $$\|q\|_{L_{n_1,n_2}}=\int_a^b(t-a)^{n_1}(b-t)^{n_2}\left|q(t)\right|dt.$$

We seek solutions of problems (1.1), (1.2) and (1.1), (1.3), respectively, in the spaces $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ and $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$.

Along with (1.1) we consider the homogeneous equation

$$u^{(n)} = \sum_{i=1}^{n} p_i(t)u^{(i-1)}, \tag{1.1$_0$}$$

and introduce the following definition.

Definition 1.1. We say that problem (1.1), (1.2) (problem (1.1), (1.3)) has the Fredholm property in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ (in the space $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$) if the unique solvability of the corresponding homogeneous problem (1.1_0), (1.2) (problem (1.1_0), (1.3)) in this space implies the unique solvability of problem (1.1), (1.2) (of problem (1.1), (1.3)) in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ (in the space $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$) for every $q \in L_{n_1,n_2}(]a,b[)$ (for every $q \in L_{0,n_0}(]a,b[)$), and for its solution the following estimate

$$\|u\|_{C^{n-1}_{n_1,n_2}} \leqslant r\|\tilde{q}\|_{C^{n-1}_{n_1,n_2}} \qquad \big(\|u\|_{C^{n-1}_{0,n_0}} \leqslant r\|\tilde{q}\|_{C^{n-1}_{0,n_0}}\big) \tag{1.5}$$

holds, where r is a positive constant independent of q and

$$\tilde{q}(t) = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds$$

$$\left(\tilde{q}(t) = \frac{1}{(n-1)!}\int_{a}^{t}(t-s)^{n-1}q(s)\,ds\right). \tag{1.6}$$

Remark 1.1. From (1.6) it is evident that

$$\|\tilde{q}\|_{C^{n-1}_{n_1,n_2}} \leqslant \rho_0\|q\|_{L_{n_1,n_2}} \qquad \big(\|\tilde{q}\|_{C^{n-1}_{0,n_0}} \leqslant \rho_0\|q\|_{L_{0,n_0}}\big),$$

where ρ_0 is a positive constant independent of q. Thus (1.5) yields the estimate

$$\|u\|_{C^{n-1}_{n_1,n_2}} \leqslant r_0\|\tilde{q}\|_{L_{n_1,n_2}} \qquad \big(\|u\|_{C^{n-1}_{n_1,n_2}} \leqslant r_0\|q\|_{L_{0,n_0}}\big),$$

where $r_0 = \rho_0 r$ is a positive constant independent of q.

In what follows problem (1.1), (1.2) is investigated in the case where the functions p_i $(i = 1,\dots,n)$ have nonintegrable singularities at the points a, t_0, and b, but

$$\int_a^b (t-a)^{n_1-n_{1i}}(b-t)^{n_2-n_{2i}}|t-t_0|^{n-i}\big|p_i(t)\big|\,dt < +\infty \quad (i = 1,\dots,n), \tag{1.7}$$

where n_{1i} and n_{2i} $(i=1,\dots,n)$ are the numbers given by equalities (1.4). As for problem (1.1), (1.3), it is considered in the case where the functions p_i $(i=1,\dots,n)$ have nonintegrable singularities only at the points a and b, and

$$\int_a^b (t-a)^{n-i}(b-t)^{n_0-n_{0i}}\left|p_i(t)\right| dt < +\infty \quad (i=1,\dots,n), \tag{1.8}$$

where

$$n_{0i} = [i+n_0-n]_+ \quad (i=1,\dots,n).$$

It is proved that in the above-mentioned cases problems (1.1), (1.2) and (1.1), (1.3) have the Fredholm property, and in a certain sense optimal conditions are found which guarantee the unique solvability of these problems.

2. Fredholm type theorems

Throughout this section, by $\overset{\circ}{C}{}^{n-1}_{n_1,n_2}(]a,b[)$ we understand the Banach space of functions $u \in C^{n-1}_{n_1,n_2}(]a,b[)$, satisfying the initial conditions

$$u^{(i-1)}(t_0) = 0 \quad (i=1,\dots,n-1)$$

with the norm $\|u\|_{\overset{\circ}{C}{}^{n-1}_{n_1,n_2}} = \|u\|_{C^{n-1}_{n_1,n_2}}$.

Lemma 2.1 in [11] implies the following lemma.

Lemma 2.1. *Let $\rho > 0$, $p_0 \in L_{n_1,n_2}(]a,b[)$ be a nonnegative function and let S be the set of $(n-1)$-times continuously differentiable functions $v :]a,b[\to \mathbb{R}$ satisfying the conditions*

$$v^{(i-1)}(t_0) = 0 \quad (i=1,\dots,n-1), \qquad \left|v^{(n-1)}(t_0)\right| \leqslant \rho \tag{2.1}$$

and

$$\left|v^{(n-1)}(t) - v^{(n-1)}(s)\right| \leqslant \int_s^t p_0(\tau)\,d\tau \quad \text{for } a < s \leqslant t < b. \tag{2.2}$$

Then S is a compact subset of the space $\overset{\circ}{C}{}^{n-1}_{n_1,n_2}(]a,b[)$.

In addition to this lemma, we need the following simple lemma.

Lemma 2.2. *Let*

$$\delta = \min\left\{\frac{t_0-a}{2}, \frac{b-t_0}{2}, 1\right\}, \qquad \gamma = (1+b-a)^{n_1+n_2}\delta^{2-2n}. \tag{2.3}$$

Then an arbitrary function $u \in \overset{\circ}{C}{}^{n-1}_{n_1,n_2}(]a,b[)$ satisfying the inequality

$$\|u\|_{\overset{\circ}{C}{}^{n-1}_{n_1,n_2}} \leqslant 1, \tag{2.4}$$

also satisfies the inequalities

$$\left|u^{(i-1)}(t)\right| \leqslant \gamma (t-a)^{-n_{1i}}(b-t)^{-n_{2i}}|t-t_0|^{n-i} \quad \textit{for } a<t<b \ (i=1,\dots,n). \tag{2.5}$$

Proof. In view of (2.4) it is clear that

$$\left|u^{(i-1)}(t)\right| \leqslant (t-a)^{-n_{1i}}(b-t)^{-n_{2i}} \quad \text{for } a<t<b \ (i=1,\dots,n), \tag{2.6}$$

and, particularly,

$$\left|u^{(n-1)}(t)\right| \leqslant (t-a)^{-n_1}(b-t)^{-n_2} \quad \text{for } a<t<b. \tag{2.7}$$

By virtue of inequality (2.7) and notation (2.3), the identities

$$u^{(i-1)}(t) = \frac{1}{(n-1-i)!}\int_{t_0}^{t}(t-s)^{n-1-i}u^{(n-1)}(s)\,ds \ (i=1,\dots,n-1)$$

imply

$$\begin{aligned}\left|u^{(i-1)}(t)\right| &\leqslant \frac{1}{(n-i)!}\delta^{-n_1-n_2}|t-t_0|^{n-i}\\ &< \frac{(b-a)^{n_{1i}+n_{2i}}}{(n-i)!}\,\delta^{-n_1-n_2}(t-a)^{-n_{1i}}(b-t)^{-n_{2i}}|t-t_0|^{n-i}\\ &< \gamma(t-a)^{-n_{1i}}(b-t)^{-n_{2i}}|t-t_0|^{n-i}\\ &\qquad \text{for } a+\delta \leqslant t \leqslant b-\delta \ (i=1,\dots,n-1).\end{aligned}$$

If together with this we take into account inequalities (2.6), then the validity of estimates (2.5) becomes obvious. □

Theorem 2.1. *If conditions* (1.7) *hold, then problem* (1.1), (1.2) *has the Fredholm property in the space* $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$.

Proof. Suppose $B = \mathring{C}^{n-1}_{n_1,n_2}(]a,b[) \times \mathbb{R}$ is a Banach space with elements $w=(u,x)$, where $u \in \mathring{C}^{n-1}_{n_1,n_2}(]a,b[)$, $x \in \mathbb{R}$, and the norm is defined in the following manner:

$$\|w\|_B = \|u\|_{\mathring{C}^{n-1}_{n_1,n_2}} + |x|.$$

For any $w=(u,x)\in B$ and $q \in L_{n_1,n_2}(]a,b[)$, we set

$$\begin{aligned} g(w)(t) &= \frac{u^{(n-1)}(t_0)+x}{(n-1)!}(t-t_0)^{n-1}\\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}\left(\sum_{i=1}^{n}p_i(s)u^{(i-1)}(s)\right)ds,\end{aligned}$$

$$\ell(w) = x + \sum_{i=1}^{n-n_1}\alpha_{1i}u^{(i-1)}(t_{1i}) + \sum_{i=1}^{n-n_2}\alpha_{2i}u^{(i-1)}(t_{2i}),$$

$$h(w)(t) = \big(g(w)(t), \ell(w)\big),$$
$$\tilde{q}(t) = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}q(s)\,ds, \qquad h_0(t) = \big(\tilde{q}(t), 0\big).$$

Then problem (1.1), (1.2) in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ is equivalent to the linear operator equation

$$w = h(w) + h_0 \tag{2.8}$$

in the space B since $w = (u,x) \in B$ is a solution of Eq. (2.8) if and only if $x = 0$ and u is a solution of problem (1.1), (1.2). As for the homogeneous equation

$$w = h(w), \tag{2.8$_0$}$$

it is equivalent to the homogeneous problem (1.1_0), (1.2).

Let

$$B_1 = \big\{w \in B\colon \ \|w\|_B \leqslant 1\big\},$$

γ be the number given by equalities (2.3),

$$\rho = 1 + (t_0-a)^{-n_1}(b-t_0)^{-n_2} + \sum_{i=1}^{n-n_1}|\alpha_{1i}|(t_{1i}-a)^{-n_{1i}}(b-t_{2i})^{-n_{2i}}$$
$$+ \sum_{i=1}^{n-n_2}|\alpha_{2i}|(t_{2i}-a)^{-n_{1i}}(b-t_{2i})^{-n_{2i}},$$

and

$$p_0(t) = \gamma\sum_{i=1}^{n}(t-a)^{-n_{1i}}(b-t)^{-n_{2i}}|t-t_0|^{n-i}\big|p_i(t)\big|.$$

Then, according to condition (1.7), $p_0 \in L_{n_1,n_2}(]a,b[)$. On the other hand, by Lemma 2.2, for any $w = (u,x) \in B_1$ the function u satisfies inequalities (2.5), and consequently, almost everywhere on $]a,b[$ we have

$$\sum_{i=1}^{n}\big|p_i(t)u^{(i-1)}(t)\big| \leqslant p_0(t).$$

If together with this we take into consideration inequalities (2.6) and (2.7), then it becomes evident that

$$\big|\ell(w)\big| \leqslant \rho,$$

and the function $v(t) = g(w)(t)$ satisfies conditions (2.1) and (2.2), i.e., $v \in S$. Thus we have shown that the linear operator h transforms the ball B_1 onto the set $S \times [-\rho,\rho]$. However, by Lemma 2.1, $S \times [-\rho,\rho]$ is a compact set of the space B. Therefore, the linear operator $h : B \to B$ is compact. By this fact and the Fredholm alternative for operator equations (see [4, Chapter XIII, §5, Theorem 1]), Eq. (2.8) is uniquely solvable if and only if Eq. (2.8_0) has only a trivial solution. Moreover, if Eq. (2.8_0) has only a trivial solution, then the operator $I - h$, where $I : B \to B$ is the identity operator, is invertible and $(I - h)^{-1} : B \to B$ is a linear bounded operator. We denote by r the norm of the operator $(I - h)^{-1}$. Then the solution $w = (u, 0)$ of Eq. (2.8) admits the estimate

$$\|w\|_B \leqslant r\|h_0\|_B = r\|\tilde{q}\|_{\tilde{C}^{n-1}_{n_1,n_2}}. \tag{2.9}$$

Since problem (1.1), (1.2) is equivalent to Eq. (2.8), it is clear that problem (1.1), (1.2) is uniquely solvable if and only if problem (1.1_0), (1.2) has only a trivial solution. Moreover, if (1.1_0), (1.2) has only a trivial solution, then for a solution u of problem (1.1), (1.2), from estimate (2.9) the estimate (1.5) follows since $w = (u, 0)$ and $\|w\|_B = \|u\|_{C^{n-1}_{n_1,n_2}}$. □

Theorem 2.2. *If conditions* (1.8) *hold, then problem* (1.1), (1.3) *has the Fredholm property in the space* $\tilde{C}^{n-1}_{0,n_0}(]a, b[)$.

Proof. Let $a_0 < a$ be an arbitrarily fixed number. Put $\alpha_0 = 0$,

$$p_i(t) = 0 \quad \text{for } a_0 \leqslant t < a \ (i = 1, \dots, n), \tag{2.10}$$

and in the interval $]a_0, b[$ consider the differential equation (1.1) with the boundary conditions

$$u^{(i-1)}(a) = 0 \quad (i = 1, \dots, n), \qquad \alpha_0 u(a_0) + \sum_{j=1}^{n-n_0} \alpha_j u^{(j-1)}(t_j) = 0. \tag{2.11}$$

Problems (1.1), (1.3) and (1.1), (2.11) are equivalent in the sense that if $q \in L_{0,n_0}(]a_0, b[)$, then the restriction of an arbitrary solution $u \in \tilde{C}^{n-1}_{0,n_0}(]a_0, b[)$ of problem (1.1), (2.11) to $]a, b[$ is a solution of problem (1.1), (1.3), and vice versa, the extension of an arbitrary solution $u \in \tilde{C}^{n-1}_{0,n_0}(]a_0, b[)$ to $]a_0, b[$ as a solution of Eq. (1.1) is a solution of problem (1.1), (2.11) in the space $\tilde{C}^{n-1}_{0,n_0}(]a_0, b[)$.

On the other hand, according to conditions (1.8) and (2.10), we have

$$\int_{a_0}^{b} (t - a)^{n-i}(b - t)^{n_0 - n_{0i}} |p_i(t)| \, dt < +\infty \quad (i = 1, \dots, n).$$

Hence, by Theorem 2.1 it follows that problem (1.1), (2.11) has the Fredholm property in the space $\tilde{C}^{n-1}_{0,n_0}(]a_0, b[)$. □

Remark 2.1. For the general boundary conditions

$$\begin{gathered} u^{(i-1)}(t_0) = 0 \quad (i = 1, \dots, n-1), \\ \sum_{j=1}^{n-n_1} \alpha_j u^{(j-1)}(a) + \sum_{j=1}^{n-n_2} \beta_j u^{(j-1)}(b) + \sum_{k=1}^{m} \sum_{j=1}^{n} \gamma_{jk} u^{(j-1)}(t_{jk}) = 0 \end{gathered} \tag{2.12}$$

and

$$u^{(i-1)}(a)=0 \quad (i=1,\dots,n-1),$$
$$\sum_{j=1}^{n-n_0}\beta_j u^{(j-1)}(b)+\sum_{k=1}^{m}\sum_{j=1}^{n}\gamma_{jk}u^{(j-1)}(\tau_{jk})=0, \tag{2.13}$$

where

$$a<t_{jk}<b, \qquad a\leqslant \tau_{jk}<b \quad (j=1,\dots,n;\ k=1,\dots,m)$$

from the proofs of Theorems 2.1 and 2.2 it is clear that if conditions (1.7) (conditions (1.8)) are satisfied, then problem (1.1), (2.12) (problem (1.1), (2.13)) has the Fredholm property in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ (in the space $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$).

3. Existence and uniqueness theorems

For Eq. (1.1_0) we consider the following two auxiliary initial conditions:

$$u^{(i-1)}(t_0)=0 \quad (i=1,\dots,n-1), \qquad u^{(n-1)}(t_0)=c; \tag{3.1}$$
$$u^{(i-1)}(a)=0 \quad (i=1,\dots,n-1), \qquad u^{(n-1)}(a)=c. \tag{3.2}$$

We first prove the following lemma.

Lemma 3.1. *If conditions* (1.7) (*conditions* (1.8)) *are fulfilled, then for any* $c\in\mathbb{R}$ *problem* (1.1_0), (3.1) (*problem* (1.1_0), (3.2)) *is uniquely solvable in the space* $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ (*in the space* $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$).

Proof. We prove the lemma only for problem (1.1_0), (3.1) since for problem (1.1_0), (3.2) it can be proved analogously. Set

$$q(t)=c\sum_{i=1}^{n}\frac{(t-t_0)^{n-i}}{(n-i)!}\,p_i(t).$$

Then, according to conditions (1.7), $q\in L_{n_1,n_2}(]a,b[)$. On the other hand, it is evident that problem (1.1_0), (3.1) is uniquely solvable in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ if and only if in the mentioned space the differential equation (1.1) has a unique solution satisfying the initial conditions

$$u^{(i-1)}(t_0)=0 \quad (i=1,\dots,n). \tag{3.3}$$

However, due to Remark 2.1, problem (1.1), (3.3) has the Fredholm property in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$. Thus, to prove the lemma, it suffices to show that the homogeneous problem (1.1_0), (3.3) in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ has only a trivial solution.

Let $u\in\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ be a solution of problem (1.1_0), (3.3). According to conditions (1.7), we have

$$v(t)\overset{\mathrm{def}}{=}\left|\int_{t_0}^{t}\left|u^{(n)}(s)\right|ds\right|<+\infty \quad \text{for } a<t<b.$$

Thus the identities

$$u^{(n-1)}(t) = \int_{t_0}^{t} u^{(n)}(s)\,ds,$$

$$u^{(i-1)}(t) = \frac{1}{(n-1-i)!}\int_{t_0}^{t} (t-s)^{n-1-i} u^{(n-1)}(s)\,ds \quad (i = 1, \dots, n-1)$$

result in

$$\left|u^{(i-1)}(t)\right| \leqslant \frac{|t-t_0|^{n-i}}{(n-i)!}\, v(t) \quad \text{for } a < t < b \ (i = 1, \dots, n).$$

On the basis of these estimates from (1.1_0) we find

$$v(t) \leqslant \left|\int_{t_0}^{t} p_0(s) v(s)\,ds\right| \quad \text{for } a < t < b,$$

where

$$p_0(t) = \sum_{i=1}^{n} \frac{|t-t_0|^{n-i}}{(n-i)!}\left|p_i(t)\right| \quad \text{and} \quad p_0 \in L_{n_1,n_2}\big(]a,b[\big).$$

Now in view of the Gronwall–Bellman lemma, the last inequality implies $v(t) \equiv 0$, and consequently, $u(t) \equiv 0$. □

Theorem 3.1. *Let conditions* (1.7) *be fulfilled and*

$$(-1)^{n-j}\alpha_{1j} \geqslant 0 \quad (j = 1, \dots, n-n_1), \qquad \alpha_{2j} \geqslant 0 \quad (j = 1, \dots, n-n_2),$$
$$\sum_{j=1}^{n-n_1} |\alpha_{1j}| + \sum_{j=1}^{n-n_2} \alpha_{2j} > 0. \tag{3.4}$$

Let, moreover, a solution u_0 of Eq. (1.1_0) *under the initial conditions*

$$u_0^{(i-1)}(t_0) = 0 \quad (i = 1, \dots, n-1), \qquad u_0^{(n-1)}(t_0) = 1 \tag{3.5}$$

satisfies the inequalities

$$(-1)^{n_1} u_0^{(n-n_1-1)}(t) > 0 \quad \text{for } a \leqslant t < t_0,$$
$$u_0^{(n-n_2-1)}(t) > 0 \quad \text{for } t_0 < t \leqslant b. \tag{3.6}$$

Then for every $q \in L_{n_1,n_2}(]a,b[)$ problem (1.1), (1.2) *is uniquely solvable in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$.*

Proof. By Theorem 2.1, it suffices to show that the homogeneous problem (1.1_0), (1.3) in the space $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$ has only the trivial solution. Assume the contrary that the mentioned problem has a nontrivial solution $u \in \tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$. Then, by Lemma 3.1, without loss of generality we may assume that $u^{(n-1)}(t_0) = 1$, and consequently,

$$u(t) \equiv u_0(t).$$

In view of this identity and conditions (3.5) and (3.6), we obtain

$$(-1)^{n-j}u^{(j-1)}(t) > 0 \quad \text{for } a \leqslant t < t_0 \ (j = 1, \dots, n - n_1),$$
$$u^{(j-1)}(t) > 0 \quad \text{for } t_0 < t \leqslant b \ (j = 1, \dots, n - n_2).$$

These inequalities and conditions (3.4) yield

$$\sum_{j=1}^{n-n_1} \alpha_{1j} u^{(j-1)}(t_{1j}) + \sum_{j=1}^{n-n_2} \alpha_{2j} u^{(j-1)}(t_{2j}) > 0.$$

But this is impossible since u is a solution of problem (1.1), (1.2). The contradiction obtained proves the theorem. □

Corollary 3.1. *Let conditions* (1.7) *and* (3.4) *hold. Let, moreover, the functions* p_i $(i = 1, \dots, n)$ *in the interval* $]a, t_0[$ *satisfy one of the following two conditions*:

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_a^{t_0} (t_0 - t)^{n-i} \big[(-1)^{n-i} p_i(t)\big]_+ \, dt \leqslant 1; \tag{3.7$_1$}$$

$$\sum_{i=1}^{n} \frac{(t_0 - t)^{n-i-1}}{(n-i-1)!} \big[(-1)^{n-i} p_i(t)\big]_+ \leqslant \lambda_{11}, \qquad \big[p_n(t)\big]_+ \leqslant \lambda_{12}, \tag{3.8$_1$}$$

and in the interval $]t_0, b[$*—one of the following two conditions*:

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_{t_0}^{b} (t - t_0)^{n-i} \big[p_i(t)\big]_- \, dt \leqslant 1; \tag{3.7$_2$}$$

$$\sum_{i=1}^{n} \frac{(t - t_0)^{n-i-1}}{(n-i-1)!} \big[p_i(t)\big]_- \leqslant \lambda_{21}, \qquad \big[p_n(t)\big]_- \leqslant \lambda_{22}, \tag{3.8$_2$}$$

where λ_{k1} *and* λ_{k2} $(k = 1, 2)$ *are nonnegative constants such that*

$$\int_0^{+\infty} \frac{ds}{\lambda_{11} + \lambda_{12} s + s^2} > t_0 - a; \tag{3.9$_1$}$$

$$\int_0^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22} s + s^2} > b - t_0. \tag{3.9$_2$}$$

Then for every $q \in L_{n_1,n_2}(]a, b[)$ *problem* (1.1), (1.2) *is uniquely solvable.*

To prove this corollary, we need the following lemma.

Lemma 3.1$_1$. *Let along with* (1.7) *either condition* (3.7$_1$) *or conditions* (3.8$_1$) *and* (3.9$_1$) *be fulfilled. Then*

$$(-1)^{n_1} u_0^{(n-n_1-1)}(t) > 0 \quad \text{for } a \leqslant t < t_0. \tag{3.10}$$

Proof. Assume that the lemma is not true. Then in view of (3.5) there exists $a_0 \in [a, t_0[$ such that

$$(-1)^{n-i} u_0^{(i-1)}(t) > 0 \quad \text{for } a_0 < t < t_0 \ (i = 1, \dots, n) \tag{3.11}$$

and

$$u_0^{(n-1)}(a_0) = 0. \tag{3.12}$$

Moreover,

$$\text{if} \quad n_1 > 0, \quad \text{then} \quad a_0 > a. \tag{3.13}$$

First we suppose that condition (3.7_1) holds. We choose $a_1 \in]a_0, t_0]$ so that

$$\rho \overset{\text{def}}{=} \max\bigl\{u_0^{(n-1)}(t)\colon\ a_0 \leqslant t \leqslant t_0\bigr\} = u_0^{(n-1)}(a_1)$$

and

$$u_0^{(n-1)}(t) < \rho \quad \text{for } a_0 \leqslant t < a_1.$$

Then, due to conditions (3.5) and (3.11), we have

$$0 < (-1)^{n-i} u_0^{(i-1)}(t) < \frac{(t_0 - t)^{n-i}}{(n-i)!}\,\rho \quad \text{for } a_0 \leqslant t < a_1 \ (i = 1, \dots, n).$$

On account of these inequalities and equality (3.12), from (1.1_0) we find

$$\rho = \int_{a_0}^{a_1} u_0^{(n)}(t)\,dt \leqslant e \sum_{i=1}^{n} \int_{a_0}^{a_1} \bigl[(-1)^{n-i} p_i(t)\bigr]_+ (-1)^{n-i} u_0^{(i-1)}(t)\,dt$$
$$< \rho \sum_{i=1}^{n} \frac{1}{(n-i)!} \int_{a_0}^{a_1} (t - t_0)^{n-i} \bigl[(-1)^{n-i} p_i(t)\bigr]_+\,dt.$$

But, according to condition (3.7_1), we obtain the contradiction $\rho < \rho$.

It remains to consider the case where conditions (3.8_1) and (3.9_1) are satisfied. For this, in view of (3.5) and (3.11), we have

$$0 < (-1)^{n-i} u_0^{(i-1)}(t) \leqslant -\frac{(t - t_0)^{n-1-i}}{(n-1-i)!}\, u_0^{(n-2)}(t) \quad \text{for } a_0 \leqslant t \leqslant t_0$$
$$(i = 1, \dots, n-1).$$

If along with this we take into account inequalities (3.8_1), then from (1.1_0) we get

$$u_0^{(n)}(t) \leqslant -\lambda_{11} u_0^{(n-2)}(t) + \lambda_1 u_0^{(n-1)}(t) \quad \text{for } a_0 \leqslant t \leqslant t_0,$$

and, consequently,

$$-\frac{u_0^{(n)}(t)}{u_0^{(n-2)}(t)} \leqslant \lambda_{11} + \lambda_1 v(t) \quad \text{for } a_0 \leqslant t < t_0,$$

where

$$v(t) = -\frac{u_0^{(n-1)}(t)}{u_0^{(n-2)}(t)},$$

and, now it follows from (3.5), (3.11), and (3.12),

$$v(t) > 0 \quad \text{for } a_0 < t < t_0, \qquad v(a_0) = 0, \qquad \lim_{t \to t_0} v(t) = +\infty. \tag{3.14}$$

On the other hand,

$$v'(t) = -\frac{u_0^{(n)}(t)}{u_0^{(n-2)}(t)} + v^2(t).$$

Therefore,

$$v'(t) \leqslant \lambda_{11} + \lambda_{12} v(t) + v^2(t) \quad \text{for } a_0 < t < t_0.$$

If we divide this inequality by $\lambda_{11} + \lambda_{12} v(t) + v^2(t)$, and then integrate from a_0 to t_0, and take into consideration (3.14), we obtain

$$\int_0^{+\infty} \frac{ds}{\lambda_{11} + \lambda_{12} s + s^2} \leqslant t_0 - a_0. \tag{3.15}$$

But this contradicts inequality (3.9_1). The contradiction obtained proves the lemma. □

Remark 3.1. If $n_1 > 0$, then by virtue of (3.13) inequality (3.15) contradicts the inequality

$$\int_0^{+\infty} \frac{ds}{\lambda_{11} + \lambda_{12} s + s^2} \geqslant t_0 - a. \tag{3.16_1}$$

Therefore, for $n_1 > 0$ condition (3.9_1) in Lemma 3.1_1 can be replaced by condition (3.16_1).

From Lemma 3.1_1, by the change of variable, we obtain the following lemma.

Lemma 3.1$_2$. *Let along with* (1.7) *either condition* (3.7_2) *or conditions* (3.8_2) *and* (3.9_2) *be fulfilled. Then*

$$u_0^{(n-n_2-1)}(t) > 0 \quad \textit{for } t_0 < t \leqslant b. \tag{3.17}$$

Proof of Corollary 3.1. By Lemmas 3.1_1 and 3.1_2, the conditions of Corollary 3.1 guarantee conditions (3.6). If now we apply Theorem 3.1, then the validity of the corollary becomes evident. □

Remark 3.2. According to Remark 3.1, if $n_1 > 0$, then condition (3.9_1) in Corollary 3.1 can be replaced by condition (3.16_1) and if $n_2 > 0$, then condition (3.9_2) can be replaced by the condition

$$\int_0^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22}s + s^2} \geqslant b - t_0. \tag{3.16_2}$$

Another corollary of Theorem 3.1 deals with the case in which $p_k(t) \equiv 0$ $(k = n - n_0 + 1, \dots, n)$, where $n_0 \in \{1, \dots, n-1\}$, i.e., the case in which Eqs. (1.1) and (1.1_0) have the forms

$$u^{(n)} = \sum_{i=1}^{n-n_0} p_i(t)u^{(i-1)} + q(t), \tag{3.18}$$

$$u^{(n)} = \sum_{i=1}^{n-n_0} p_i(t)u^{(i-1)}, \tag{3.18_0}$$

respectively.

Corollary 3.2. *Let* $n_i \in \{1, \dots, n_0\}$ $(i = 1, 2)$ *and along with* (3.4) *the conditions*

$$\int_a^b (t-a)^{n_1}(b-t)^{n_2}|t-t_0|^{n-i}\big|p_i(t)\big|\,dt < +\infty \quad (i = 1, \dots, n-n_0), \tag{3.19}$$

$$\sum_{i=1}^{n-n_0} \frac{1}{(n-i)!} \int_a^{t_0} (t_0-t)^{n-i}(t-a)^{n_1}\big[(-1)^{n-i}p_i(t)\big]_+\,dt \leqslant (t_0-a)^{n_1}, \tag{3.20}$$

$$\sum_{i=1}^{n-n_0} \frac{1}{(n-i)!} \int_{t_0}^b (t-t_0)^{n-i}(b-t)^{n_2}\big[p_i(t)\big]_-\,dt \leqslant (b-t_0)^{n_2} \tag{3.21}$$

be fulfilled. Then for every $q \in L_{n_1,n_2}(]a,b[)$ *problem* (1.1), (1.2) *is uniquely solvable in the space* $\tilde{C}^{n-1}_{n_1,n_2}(]a,b[)$.

Proof. By Theorem 3.1, it suffices to show that a solution u_0 of problem (3.18_0), (3.5) satisfies inequalities (3.10) and (3.17). We give only the proof of inequality (3.10) since inequality (3.17) can be proved analogously.

Assume the contrary that inequality (3.10) is violated. Then there exists $a_0 \in [a, t_0[$ such that

$$(-1)^{n-i}u_0^{(i-1)}(t) > 0 \quad \text{for } a_0 < t < t_0 \ (i = 1, \dots, n-n_1) \tag{3.22}$$

and

$$u_0^{(n-n_1-1)}(a_0) = 0. \tag{3.23}$$

Put

$$m = n_1 + 1, \qquad v(t) = (-1)^{m-1} u_0^{(n-m)}(t), \qquad q_0(t) = \sum_{i=1}^{n-n_0} p_i(t) u_0^{(i-1)}(t).$$

Then, in view of conditions (3.5) and (3.23), the function v is a solution of the problem

$$v^{(m)} = (-1)^{m-1} q_0(t),$$
$$v(a_0) = 0, \qquad v^{(i-1)}(t_0) = 0 \quad (i = 1, \dots, m-1),$$

satisfying the conditions

$$v(t) > 0 \quad \text{for } a_0 < t < t_0, \qquad \lim_{t \to t_0} \frac{(m-1)!\,v(t)}{(t_0 - t)^{m-1}} = 1.$$

Thus

$$1 \leqslant \rho = \sup\left\{ \frac{(m-1)!\,v(t)}{(t_0 - t)^{m-1}} \colon a_0 \leqslant t < t_0 \right\} < +\infty,$$

and there exists $a_1 \in\,]a_0, t_0]$ such that

$$v(t) < \frac{(t_0 - t)^{m-1}}{(m-1)!} \rho \quad \text{for } a_0 \leqslant t < a_1 \tag{3.24}$$

and

$$v(a_1) = \frac{(t_0 - a_1)^{m-1}}{(m-1)!} \rho. \tag{3.25}$$

On the other hand, according to (3.5) and (3.22), we have

$$q_0(t) \leqslant \sum_{i=1}^{n-n_0} \big[(-1)^{n-i} p_i(t)\big]_+ \big|u_0^{(i-1)}(t)\big| \quad \text{for } a_0 < t < t_0. \tag{3.26}$$

Moreover, if $n_1 < n - i$, then

$$\big|u_0^{(i-1)}(t)\big| = \frac{1}{(n - n_1 - 1 - i)!} \int_t^{t_0} (s - t)^{n - n_1 - 1 - i} v(s)\, ds \quad (i = 1, \dots, n - n_1 - 1).$$

The last identities and inequalities (3.24) imply

$$\big|u_0^{(i-1)}(t)\big| \leqslant \frac{(t_0 - t)^{n-i}}{(n-i)!} \rho \quad \text{for } a_1 \leqslant t \leqslant t_0,$$
$$\big|u_0^{(i-1)}(t)\big| < \frac{(t_0 - t)^{n-i}}{(n-i)!} \rho \quad \text{for } a_0 \leqslant t < a_1 \ (i = 1, \dots, n_0). \tag{3.27}$$

According to Green's function and equality (3.25), the representation

$$\frac{(t_0 - a_1)^{m-1}}{(m-1)!} \rho = v(a_1) = \int_{a_0}^{t_0} g(a_1, s) q_0(s)\, ds \tag{3.28}$$

is valid, where

$$0 < g(a_1, s) = \frac{1}{(m-1)!}\left(\frac{s-a_0}{t_0-a_0}\right)^{m-1}(t_0-a_1)^{m-1}$$
$$\leqslant \frac{1}{(m-1)!}\left(\frac{s-a}{t_0-a}\right)^{n_1}(t_0-a_1)^{m-1} \quad \text{for } a_0 < s \leqslant a_1,$$
$$0 < g(a_1, s) = \frac{1}{(m-1)!}\left[\left(\frac{s-a_0}{t_0-a_0}\right)^{m-1} - \left(\frac{s-a_1}{t_0-a_1}\right)^{m-1}\right](t_0-a_1)^{m-1}$$
$$< \frac{1}{(m-1)!}\left(\frac{s-a}{t_0-a}\right)^{n_1}(t_0-a_1)^{m-1} \quad \text{for } a_1 < s \leqslant t_0.$$

If along with the last two inequalities we take into account inequalities (3.19), (3.26), and (3.27), then from (3.28) we find

$$\rho < \rho(t_0-a)^{-n_1}\sum_{i=1}^{n-n_0}\frac{1}{(n-i)!}\int_a^{t_0}(t_0-t)^{n-i}(t-a_1)^{n_1}\big[(-1)^{n-i}p_i(t)\big]_+\,dt \leqslant \rho.$$

The obtained contradiction proves the validity of inequality (3.10). □

As mentioned above, problem (1.1), (1.3) is equivalent to problem (1.1), (2.10), where $\alpha_0 = 0$, $a_0 < a$, and the functions p_i $(i = 1,\dots,n)$ are extended to $]a_0, b[$ by equalities (2.10). In view of this fact, from Theorem 3.1 and its corollaries the following propositions on the unique solvability of problems (1.1), (1.3) and (3.18), (1.3) follow rather easily.

Theorem 3.2. *Let conditions* (1.8) *and*

$$\alpha_j \geqslant 0 \quad (j = 1,\dots,n_0), \qquad \sum_{j=1}^{n-n_0}\alpha_j > 0 \tag{3.29}$$

hold. Let, moreover, a solution u_0 of Eq. (1.1_0), *together with the initial conditions*

$$u_0^{(i-1)}(a) = 0 \quad (i = 1,\dots,n-1), \qquad u_0^{(n-1)}(a_0) = 1,$$

satisfy the inequality

$$u_0^{(n-n_0-1)}(t) > 0 \quad \textit{for } a < t \leqslant b.$$

Then for every $q \in L_{0,n_0}(]a,b[)$ problem (1.1), (1.3) *is uniquely solvable in the space* $\tilde{C}^{n-1}_{0,n_0}(]a,b[)$.

Corollary 3.3. *Let along with* (1.8) *and* (3.29) *either the condition*

$$\sum_{i=1}^{n}\frac{1}{(n-i)!}\int_a^b (t-a)^{n-i}\big[p_i(t)\big]_-\,dt \leqslant 1, \tag{3.30}$$

or the condition

$$\sum_{i=1}^{n-1} \frac{(t-a)^{n-i-1}}{(n-i-1)!} \big[p_i(t)\big]_- \leqslant \lambda_1, \qquad \big[p_n(t)\big]_- \leqslant \lambda_2 \quad \textit{for } a < t < b \tag{3.31}$$

holds, where λ_i $(i = 1, 2)$ *are nonnegative constants such that*

$$\int_0^{+\infty} \frac{ds}{\lambda_1 + \lambda_2 s + s^2} > b - a. \tag{3.32}$$

Then for every $q \in L_{0,n_0}(]a, b[)$ *problem* (1.1), (1.3) *is uniquely solvable in the space* $\tilde{C}^{n-1}_{0,n_0}(]a, b[)$.

Corollary 3.4. *If along with* (3.29) *the conditions*

$$\int_a^b (t-a)^{n-i}(b-t)^{n_0}\big|p_i(t)\big|\,dt < +\infty \quad (i = 1, \ldots, n - n_0), \tag{3.33}$$

$$\sum_{i=1}^{n-n_0} \frac{1}{(n-i)!} \int_a^b (t-a)^{n-i}(b-t)^{n_0}\big[p_i(t)\big]_-\,dt \leqslant (b-a)^{n_0} \tag{3.34}$$

are satisfied, then for every $q \in L_{0,n_0}(]a, b[)$ *problem* (3.18), (1.3) *is uniquely solvable in the space* $\tilde{C}^{n-1}_{0,n_0}(]a, b[)$.

Remark 3.3. For $n_0 > 0$ condition (3.32) in Corollary 3.3 can be replaced by the condition

$$\int_0^{+\infty} \frac{ds}{\lambda_1 + \lambda_2 s + s^2} \geqslant b - a. \tag{3.35}$$

Remark 3.4. Corollaries 3.3 and 3.4 are the generalizations (for the singular problems (1.1), (1.3) and (3.18), (1.3)) of the Vallée-Poussin [26] and Hartman–Wintner [3] well-known results on the unique solvability of two-point boundary value problems for second order linear differential equations with continuous coefficients.

Remark 3.5. The presence of the Fredholm property for problems (1.1), (1.2) and (1.1), (1.3) and the unique solvability of these problems in the spaces $\tilde{C}^{n-1}_{n_1,n_2}(]a, b[)$ and $\tilde{C}^{n-1}_{0,n_0}(]a, b[)$, respectively, do not guarantee the existence of a solution of Eq. (1.1) in the mentioned spaces, satisfying the boundary conditions

$$u^{(i-1)}(t_0) = c_i \quad (i = 1, \ldots, n-1),$$
$$\sum_{j=1}^{n-n_1} \alpha_{1j} u^{(j-1)}(t_{1j}) + \sum_{j=1}^{n-n_2} \alpha_{2j} u^{(j-1)}(t_{2j}) = 0,$$

or

$$u^{(i-1)}(a) = c_i \quad (i = 1, \ldots, n-1), \qquad \sum_{j=1}^{n-n_0} \alpha_j u^{(j-1)}(t_j) = 0,$$

where $\sum_{i=1}^{n} |c_i| \neq 0$. Indeed, as examples we consider the boundary value problems

$$u^{(n)} = \sum_{i=1}^{n} \frac{p_{0i}(t)\,\mathrm{sign}(t-t_0)^{n-i+1}}{(|t-t_0|(t-a)(b-t))^{n-i+\varepsilon}} u^{(i-1)} + \frac{q_0(t)}{((t-a)(b-t))^{n-1+\varepsilon}}, \tag{3.36}$$

$$u^{(i-1)}(t_0) = c_i \quad (i = 1, \ldots, n-1), \qquad (-1)^{n-1} u(a) + u(b) = 0, \tag{3.37}$$

and

$$u^{(n)} = \sum_{i=1}^{n} \frac{p_{0i}(t)}{((t-a)(b-t))^{n-i+\varepsilon}}\, u^{(i-1)} + \frac{q_0(t)}{((t-a)(b-t))^{n-1+\varepsilon}}, \tag{3.38}$$

$$u^{(i-1)}(a) = c_i \quad (i = 1, \ldots, n-1), \qquad u(b) = 0, \tag{3.39}$$

where $p_{0i} : [a,b] \to]0, +\infty[$ $(i = 1, \ldots, n)$ and $q_0 : [a,b] \to \mathbb{R}$ are continuous functions, and $\varepsilon \in [0, 1[$. If $c_i = 0$ $(i = 1, \ldots, n)$, then according to Theorem 2.1 and Corollary 3.1 (according to Theorem 2.2 and Corollary 3.3) problem (3.36), (3.37) (problem (3.38), (3.39)) has the Fredholm property and is uniquely solvable in the space $\tilde{C}^{n-1}_{n-1,n-1}(]a,b[)$ (in the space $\tilde{C}^{n-1}_{0,n-1}(]a,b[)$). On the other hand, it is evident that if $c_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $\sum_{i=1}^{n-1} c_i > 0$, then problem (3.36), (3.37) (problem (3.38), (3.39)) does not have a solution in the mentioned space.

4. Examples

In this section, we give examples verifying the optimality of conditions in Corollaries 3.1–3.4 guaranteeing the unique solvability of problems (1.1), (1.2) and (1.1), (1.3).

Example 4.1. Let $t_0 \in]a, b[$, $\varepsilon \in]0, 1[$, $k \in \{1, \ldots, n-1\}$,

$$w(t) = \frac{1}{(n-2)!} \int_{t_0}^{t} (t-s)^{n-2} \left(\frac{b-s}{b-t_0}\right)^{\varepsilon} ds \quad \text{for } t_0 \leqslant t \leqslant b,$$

$$p_k(t) = \begin{cases} 0 & \text{for } a < t \leqslant t_0, \\ -\varepsilon(b-t)^{\varepsilon-1}/(b-t_0)^{\varepsilon} w^{(k-1)}(t) & \text{for } t_0 < t < b, \end{cases}$$

$$p_i(t) = 0 \quad \text{for } a < t < b \ (i \neq k;\ i = 1, \ldots, n).$$

Then

$$w^{(n-2)}(t) = \int_{t_0}^{t} \left(\frac{b-s}{b-t_0}\right)^{\varepsilon} ds = \frac{b-t_0}{1+\varepsilon}\left(1 - \left(\frac{b-t}{b-t_0}\right)^{1+\varepsilon}\right) > \frac{t-t_0}{1+\varepsilon}$$
$$\text{for } t_0 < t < b,$$

$$\left|p_k(t)\right| = -p_k(t) < \varepsilon(1+\varepsilon)(n-k)!(t-t_0)^{k-n}(b-t_0)^{-\varepsilon}(b-t)^{\varepsilon-1} \quad \text{for } t_0 < t < b.$$

Therefore, evidently, conditions (1.7) are satisfied. Moreover, in the interval $]a, t_0[$ both conditions (3.7_1) and (3.8_1) hold, where $\lambda_{11} = \lambda_{12} = 0$, and in the interval $]t_0, b[$ the inequality

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_{t_0}^{b} (t-t_0)^{n-i} \big[p_i(t)\big]_- \, dt < 1 + \varepsilon \tag{4.1}$$

is fulfilled instead of (3.7_2). Nevertheless the homogeneous problem (1.1_0) under the boundary conditions

$$u^{(i-1)}(t_0) = 0 \quad (i = 1, \ldots, n-1), \qquad u^{(n-1)}(b) = 0 \tag{4.2}$$

has the nontrivial solution

$$u(t) = \begin{cases} \frac{(t-t_0)^{n-1}}{(n-1)!} & \text{for } a \leqslant t \leqslant t_0, \\ w(t) & \text{for } t_0 < t \leqslant b, \end{cases}$$

in the space $\tilde{C}^{n-1}_{0,0}(]a, b[)$. The constructed example shows that condition (3.7_2) in Corollary 3.1 cannot be replaced by condition (4.1) no matter how small $\varepsilon > 0$. In view of this example, it also becomes evident that condition (3.7_1) in Corollary 3.1 (condition (3.30) in Corollary 3.3) cannot be replaced by the condition

$$\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_{a}^{t_0} (t_0-t)^{n-i} \big[(-1)^{n-i} p_i(t)\big]_+ \, dt \leqslant 1 + \varepsilon,$$

$$\left(\sum_{i=1}^{n} \frac{1}{(n-i)!} \int_{a}^{b} (t-a)^{n-i} \big[p_i(t)\big]_- \, dt \leqslant 1 + \varepsilon \right).$$

Example 4.2. Suppose $t_0 \in]a, b[$, and λ_{21}, λ_{22} are positive constants such that

$$\int_{0}^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22} s + s^2} = b - t_0. \tag{4.3}$$

Put

$$p_{n-1}(t) = \begin{cases} 0 & \text{for } a \leqslant t \leqslant t_0, \\ -\lambda_{21} & \text{for } t_0 < t \leqslant b, \end{cases} \qquad p_n(t) = \begin{cases} 0 & \text{for } a \leqslant t \leqslant t_0, \\ -\lambda_{22} & \text{for } t_0 < t \leqslant b, \end{cases}$$

and

$$p_i(t) = 0 \quad \text{for } a \leqslant t \leqslant b \ (i = 1, \ldots, n-2) \quad \text{if } n > 3.$$

Then the functions p_i $(i = 1, \ldots, n)$ satisfy conditions (3.7_1) and (3.8_1) in the interval $]a, t_0[$, where $\lambda_{11} = \lambda_{12} = 0$, and conditions (3.8_2) in the interval $]t_0, b[$, where λ_{21} and λ_{22} satisfy equality (4.3) instead of inequality (3.9_2). We show that the homogeneous problem

(1.1_0), (4.2) in the space $\tilde{C}^{n-1}_{0,0}(]a,b[)$ has a nontrivial solution. Indeed, let u be a solution of Eq. (1.1_0) satisfying the initial conditions

$$u^{(i-1)}(t_0) = 0 \quad (i = 1, \dots, n-1), \qquad u^{(n-1)}(t_0) = 1,$$

and let

$$b_0 = \sup\bigl\{t \in]t_0, b[:\ u^{(n-1)}(s) > 0 \text{ for } t_0 < s < t\bigr\}.$$

Set

$$v(t) = \frac{u^{(n-1)}(t)}{u^{(n-2)}(t)} \quad \text{for } t_0 < t < b_0.$$

Then $v(t) > 0$ for $t_0 < t < b$, $v(t) \to +\infty$ as $t \to t_0$, and

$$v'(t) = -\lambda_{21} - \lambda_{22} v(t) - v^2(t) \quad \text{for } t_0 < t < b_0.$$

Therefore,

$$\int\limits_{v(b_0)}^{+\infty} \frac{ds}{\lambda_{21} + \lambda_{22} s + s^2} = b_0 - t_0.$$

Hence, in view of equality (4.3), it follows that $b_0 = b$ and $v(b) = 0$. Thus $u^{(n-1)}(b) = 0$ and, consequently, $u \in \tilde{C}^{n-1}_{0,0}(]a,b[)$ is a nontrivial solution of problem (1.1_0), (4.2). The constructed example shows that if $n_2 = 0$ ($n_1 = 0$), then condition (3.9_2) (condition (3.9_1)) in Corollary 3.1 cannot be replaced by condition (3.16_1) (by condition (3.16_2)). This example also shows that if $n_0 = 0$, then condition (3.32) in Corollary 3.3 cannot be replaced by condition (3.35).

Example 4.3. Let $\varepsilon \in]0,1[$, $\varepsilon_0 = \varepsilon/2$, $t_0 \in]a,b[$, $b_0 = (t_0 + b)/2$,

$$p_{n-1}(t) = \begin{cases} 0 \quad \text{for } a \leqslant t \leqslant t_0, \\ -\varepsilon_0(1+\varepsilon_0)|t-b_0|^{\varepsilon_0 - 1}\bigl[(b-b_0)^{1+\varepsilon_0} - |t-b_0|^{1+\varepsilon_0}\bigr]^{-1} \\ \quad \text{for } t_0 < t < b,\ t \neq b_0, \end{cases}$$

and

$$p_i(t) = 0 \quad \text{for } a \leqslant t \leqslant b \ (i = 1, \dots, n-2) \quad \text{if } n > 2.$$

We consider Eq. (3.18_0), where $n_0 = 1$, with the boundary conditions

$$u^{(i-1)}(t_0) = 0 \quad (i = 1, \dots, n-1), \qquad u^{(n-2)}(b) = 0. \tag{4.4}$$

In view of the definition of p_{n-1} we have

$$\bigl|p_n(t)\bigr| = -p_{n-1}(t) \quad \text{for } t_0 < t < b,\ t \neq b_0,$$

$$p_{n-1}(t) = p_{n-1}(2b_0 - t) \quad \text{for } b_0 < t < b,$$

$$\bigl|p_{n-1}(t)\bigr| < \varepsilon_0(1+\varepsilon_0)(b-b_0)^{-\varepsilon_0}(t-b_0)^{\varepsilon_0 - 1}(b-t)^{-1} \quad \text{for } b_0 < t < b.$$

Thus

$$\int_{t_0}^{b} (t-t_0)(b-t)\big[p_{n-1}(t)\big]_- \,dt = 2\int_{b_0}^{b} (t-t_0)(b-t)\big|p_{n-1}(t)\big|\,dt$$
$$< 2\varepsilon_0(1+\varepsilon_0)(b-b_0)^{-\varepsilon_0}\int_{b_0}^{b} (t-t_0)(t-b_0)^{\varepsilon_0-1}\,dt$$
$$= 2(1+\varepsilon_0)\left(b-t_0-\frac{b-b_0}{1+\varepsilon_0}\right)$$
$$= (1+2\varepsilon_0)(b-t_0) = (1+\varepsilon)(b-t_0).$$

Therefore, the functions p_i $(i=1,\ldots,n-1)$ satisfy conditions (3.19) and (3.20), where $n_1 = n_2 = n_0$, and instead of (3.21) the condition

$$\sum_{i=1}^{n-n_0} \frac{1}{(n-i)!}\int_{t_0}^{b} (t-t_0)^{n-i}(b-t)^{n_2}\big[p_i(t)\big]_-\,dt \leqslant (1+\varepsilon)(b-t_0)^{n_2} \tag{4.5}$$

holds. We show that the problem (3.18_0), (4.4) has a nontrivial solution. Indeed, suppose

$$w(t) = \begin{cases} (1+\varepsilon_0)(b_0-t_0)^{\varepsilon_0}(t-t_0) & \text{for } a \leqslant t \leqslant t_0, \\ (b-b_0)^{1+\varepsilon_0} - |t-b_0|^{1+\varepsilon_0} & \text{for } t_0 < t \leqslant b. \end{cases}$$

Then w is a solution of the problem

$$w'' = p_{n-1}(t)w; \qquad w(t_0) = 0, \qquad w(b) = 0.$$

We put

$$u(t) = \frac{1}{(n-3)!}\int_{t_0}^{t} (t-s)^{n-3} w(s)\,ds$$

if $n \geqslant 3$, and $u(t) = w(t)$ if $n = 2$. Obviously, $u \in \tilde{C}_{1,1}^{n-1}(]a,b[)$ is a nontrivial solution of problem (3.18_0), (4.4).

The constructed example shows that condition (3.21) in Corollary 3.2 cannot be replaced by condition (4.5) no matter how small $\varepsilon > 0$. Due to this example, it is also evident that condition (3.20) in Corollary 3.2 and condition (3.34) in Corollary 3.4 are optimal as well.

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